Journal of Nonlinear Mathematical Physics Volume 10, Number 1 (2003), 110–135 ARTICLE

On CP^1 and CP^2 Maps and Weierstrass Representations for Surfaces Immersed into Multi-Dimensional Euclidean Spaces

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Received June 07, 2002; Accepted July 24, 2002

Abstract

An extension of the classic Enneper–Weierstrass representation for conformally parametrised surfaces in multi-dimensional spaces is presented. This is based on low dimensional CP^1 and CP^2 sigma models which allow the study of the constant mean curvature (CMC) surfaces immersed into Euclidean 3- and 8-dimensional spaces, respectively. Relations of Weierstrass type systems to the equations of these sigma models are established. In particular, it is demonstrated that the generalised Weierstrass representation can admit different CMC-surfaces in $\mathbb{R}^3$ which have globally the same Gauss map. A new procedure for constructing CMC-surfaces in $\mathbb{R}^n$ is presented and illustrated in some explicit examples.

1 Introduction

In this paper we study two-dimensional surfaces conformally immersed into multi dimensional spaces with Euclidean metric. We present explicit formulae for the position vector $X : \mathcal{D} \in \mathbb{C} \to \mathbb{R}^3$ of a surface for which X satisfies the Gauss–Weingarten and Gauss–Codacci equations. Such formulae describing minimal surfaces (i.e. zero mean curvature $H = 0$) imbedded in three-dimensional space were first formulated by Enneper [1] and Weierstrass [2] about one and half century ago. These authors considered two holomorphic functions $\psi(z)$ and $\phi(z)$ of a complex variable $z \in \mathbb{C}$ and introduced a three component complex vector valued function $w = (w_1, w_2, w_3) : \mathcal{D} \to \mathbb{C}^3$ which is required to satisfy the following linear differential equations

$$\partial w_1 = \frac{1}{2}\left(\psi^2 - \phi^2\right), \qquad \partial w_2 = \frac{i}{2}\left(\psi^2 + \phi^2\right), \qquad \partial w_3 = -\psi\phi,$$
$$\bar{\partial}\psi = 0, \qquad \bar{\partial}\phi = 0, \tag{1.1}$$

where the derivatives are abbreviated $\partial = \partial/\partial z$ and the bar denotes the complex congugation. Then they showed that the real vector valued functions

$$X = \left(\mathrm{Re}\int_0^z \frac{1}{2}\left(\psi^2 - \phi^2\right)\,dz', \mathrm{Re}\int_0^z \frac{i}{2}\left(\psi^2 + \phi^2\right)\,dz', -\mathrm{Re}\int_0^z \psi\phi\,dz'\right) \tag{1.2}$$

can be considered as components of a position vector of a minimal surface, immersed into $\mathbb{R}^3$, with the conformal metric

$$ds^2 = \left(|\psi|^2 + |\phi|^2\right)^2\,dz\,d\bar{z}, \tag{1.3}$$

where z and $\bar{z}$ are local coordinates on $\mathcal{D}$ and the coordinate lines $z = \mathrm{const}$, and $\bar{z} = \mathrm{const}$ describe geodesics on this surface. Since then this idea has been developed by many authors, for a review of the subject see e.g. [3, 4, 5] and references therein. The theory of minimal, or in general, constant mean curvature (CMC)-surfaces plays an important role in several applications to problems arising both in mathematics and in physics. In particular, many interesting applications can be found in such diverse areas of physics as: the fields of two-dimensional gravity [6, 7], string theory [8], quantum field theory [9, 6], statistical physics [10, 11], fluid dynamics [12], theory of fluid membranes [13, 10]. One interesting application involves the Canham–Helfrich membrane model [14, 15]. This model can explain some basic features and equilibrium shapes both for biological membranes and liquid interfaces [16].

Our approach involves modifying the original Enneper–Weierstrass representation (1.2) by adding to it extra terms. For this purpose it is convenient to exploit the connection between Weierstrass systems, CP^1 and CP^2 sigma model equations, and their Lax representations. We demonstrate that, through these links, conformal immersion of CMC-surfaces into 3- and 8-dimensional spaces can be formulated. We show that a large classes of solutions of the Weierstrass system can be obtained and, consequently, can provide new classes of conformally parametrised CMC-surfaces in multi-dimensional spaces.

The paper is organized as follows. In Section 2, we rederive the classical Enneper–Weierstrass representation for minimal surfaces immersed into $\mathbb{R}^3$. In the next section we describe, in detail, the generalised Weierstrass formulae for CMC-surfaces into $\mathbb{R}^3$ in the context of the CP^1 sigma model and discuss some geometric aspects of CP^1 maps. The following section deals with CP^2 maps and the corresponding Weierstrass representation for conformally parametrised surfaces immersed into $\mathbb{R}^8$. It also presents some geometric characteristics of CMC-surfaces. In Section 5 our theoretical considerations are illustrated by explicit examples and new interesting CMC-surfaces are found. The last section presents final remarks and mentions possible future developments.

2 The Enneper–Weierstrass formulae for minimal surfaces in $\mathbb{R}^3$

Let $\mathbb{M}^2$ be a smooth orientable surface in 3-dimensional Euclidean space $\mathbb{R}^3$. The surface $\mathbb{M}^2$ is described by a real vector-valued function

$$X = (X_1, X_2, X_3) : D \to \mathbb{R}^3, \tag{2.1}$$

where D is a region in the complex plane $\mathbb{C}$. The metric is assumed to be conformally flat

$$ds^2 = e^{2u}\, dz\, d\bar{z} \tag{2.2}$$

for any real valued function u of z and $\bar{z}$. The conformal parametrisation of the surface $\mathbb{M}^2$ implies the following normalization of the position vector $X(z, \bar{z})$

$$(\partial X, \partial X) = 0, \qquad (\partial X, \bar{\partial} X) = \frac{1}{2} e^{2u}, \tag{2.3}$$

where the brackets $(\cdot, \cdot)$ denote the standard scalar product in $\mathbb{R}^3$. The tangent vectors ∂X and $\bar{\partial} X$ and the real unit normal vector N on the surface $\mathbb{M}^2$ satisfy the obvious relations

$$(\partial X, N) = 0, \qquad (N, N) = 1. \tag{2.4}$$

Equations of a moving complex frame $\xi = (\partial X, \bar{\partial} X, N)^T$ satisfy the following Gauss–Weingarten equations (see e.g. [5])

$$\partial \xi = U \xi, \qquad \bar{\partial} \xi = V \xi, \tag{2.5}$$

where 3×3 matrices U and V have the form

$$U = \begin{pmatrix} 2\partial u & 0 & J \\ 0 & 0 & \frac{1}{2} H e^{2u} \\ -H & -2e^{-2u} J & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 0 & \frac{1}{2} H e^{2u} \\ 0 & 2\bar{\partial} u & \bar{J} \\ -2e^{-2u} \bar{J} & -H & 0 \end{pmatrix}, \tag{2.6}$$

and the following notation has been introduced

$$J = (\partial^2 X, N), \qquad H = 2e^{-2u} (\partial \bar{\partial} X, N). \tag{2.7}$$

Formulae (2.5) are compatible with the scalar products (2.3) and (2.4). From (2.5) and (2.6) we can derive the equation for the unit normal vector N

$$\partial \bar{\partial} N + (\partial N, \bar{\partial} N) N + \bar{\partial} H \partial X + \partial H \bar{\partial} X = 0. \tag{2.8}$$

The corresponding Gauss–Codazzi equations of the conformally parametrised surface $\mathbb{M}^2 \subset \mathbb{R}^3$ are the compatibility conditions of equations (2.5) and have the following form

$$\partial \bar{\partial} u + \frac{1}{4} H^2 e^{2u} - 2|J|^2 e^{-2u} = 0, \tag{2.9}$$

$$\bar{\partial} J = \frac{1}{2} \partial H e^{2u}, \qquad \partial \bar{J} = \frac{1}{2} \bar{\partial} H e^{2u}. \tag{2.10}$$

The aim of this section is to rederive the original Enneper–Weierstrass formulae [1, 2] for inducing minimal surfaces in $\mathbb{R}^3$. For surfaces with $H = 0$ the formulae given above simplify considerably. We focus our attention on the construction of the explicit formula for the position vector $X(z, \bar{z})$ of conformally parametrised surfaces into $\mathbb{R}^3$ for which equations (2.3), (2.9) and (2.10) are fulfilled.

For computational purposes, it is useful to examine equations (2.3), (2.9) and (2.10) in terms of a two-component object which, in fact, is a spinor, but its spinorial nature is not relevant to our discussion: $\phi = (\psi_1, \psi_2)^T \in \mathbb{C}^2$. We show that, by quadratures, we can determine the coordinates of the position vector $X(z, \bar{z})$ in terms of the components of ϕ satisfying equations (2.3) and (2.9)–(2.10).

Let us consider the complex vector $\vec{w}$ in $\mathbb{C}^3$ equal to one of the tangent vectors, say, ∂X

$$\vec{w} = (w_1, w_2, w_3) = \partial X, \qquad w_i \in \mathbb{C}, \quad i = 1, 2, 3, \tag{2.11}$$

the 2 by 2 traceless matrix

$$w = \begin{pmatrix} w_3 & w_1 - iw_2 \\ w_1 + iw_2 & -w_3 \end{pmatrix}, \qquad \operatorname{tr} w = 0, \tag{2.12}$$

and the map

$$\vec{w} : \mathbb{C} \to w = w_i \sigma_i \in sl(2, \mathbb{C}), \tag{2.13}$$

where σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.14}$$

The map (2.13) satisfies

$$\vec{w}^2 = -\det w. \tag{2.15}$$

From (2.15), the determinant of the matrix w vanishes if and only if the vector $\vec{w}$ is null, which coincides with the first condition in (2.3). Hence, using (2.12), we can express uniquely the null vector $\vec{w}$ in terms of the complex two-component vector ϕ as follows

$$w_1 = \frac{1}{2}\left(\psi_1^2 - \psi_2^2\right), \qquad w_2 = \frac{i}{2}\left(\psi_1^2 + \psi_2^2\right), \qquad w_3 = -\psi_1\psi_2. \tag{2.16}$$

From the assumption (2.11) that the null vector $\vec{w}$ is equal to the tangent vector ∂X, we can express ∂X in terms of ψ_1 and ψ_2 as follows:

$$\partial X_1 = \frac{1}{2}\left(\psi_1^2 - \psi_2^2\right), \qquad \partial X_2 = \frac{i}{2}(\psi_1^2 + \psi_2^2), \qquad \partial X_3 = -\psi_1\psi_2, \tag{2.17}$$

which coincide with expression (1.1). The Enneper–Weierstrass representation for surfaces in $\mathbb{R}^3$ are obtained under the additional assumption that ψ_1 and ψ_2 are arbitrary holomorphic functions of the complex variable $z \in \mathbb{C}$. Then, integrating equations (2.17) and taking into account the reality condition of the position vector

$$X(z, \bar{z}) = \bar{X}(z, \bar{z}), \tag{2.18}$$

we obtain the following representation [2]

$$\begin{aligned}
X_1 &= \frac{1}{2}\int_0^z \left(\psi_1^2 - \psi_2^2\right) dz' + \frac{1}{2}\int_0^{\bar{z}} \left(\bar{\psi}_1^2 - \bar{\psi}_2^2\right) dz', \\
X_2 &= \frac{i}{2}\int_0^z \left(\psi_1^2 + \psi_2^2\right) dz' - \frac{i}{2}\int_0^{\bar{z}} \left(\bar{\psi}_1^2 + \bar{\psi}_2^2\right) d\bar{z}', \\
X_3 &= -\int_0^z \psi_1\psi_2\, dz' - \int_0^{\bar{z}} \bar{\psi}_1\bar{\psi}_2\, d\bar{z},
\end{aligned} \tag{2.19}$$

which, in fact, is equivalent to (1.2). Next, from (2.19) and invoking the second condition (2.3) we find that

$$u = \ln\left(|\psi_1|^2 + |\psi_2|^2\right). \tag{2.20}$$

Substituting (2.20) into the Gauss–Codazzi equations (2.9)–(2.10), we obtain

$$\partial\bar{\partial}\ln p^2 - 2|J|^2 p^{-2} = 0, \tag{2.21}$$

where

$$p = |\psi_1|^2 + |\psi_2|^2. \tag{2.22}$$

By virtue of (2.19), we find that J and H defined by (2.7), when expressed in terms of ψ_1 and ψ_2 become

$$J = \psi_1\partial\psi_2 - \psi_2\partial\psi_1, \qquad H = 0 \tag{2.23}$$

and J is analytic, i.e. $\bar{\partial}J = 0$.

Note that the direction of $\phi = (\psi_1, \psi_2)^T$ is arbitrary, but its length is fixed by (2.21). Note also that after the change of variable $\varphi = 2\ln p$ equation (2.21) becomes

$$\partial\bar{\partial}\varphi = 2|J|^2 e^{-\varphi}, \qquad \bar{\partial}J = 0. \tag{2.24}$$

3 The generalised Weierstrass formulae for CMC-surfaces in $\mathbb{R}^3$

The Wierstrass–Enneper formulae for inducing minimal surfaces, and their generalisations, has been studied for a long time by many authors (e.g. [3, 17, 18] and references therein). This topic has most recently been treated by B Konopelchenko and I Taimanov [19]. In this paper, Konopelchenko and Taimanov, established a direct connection between certain classes of CMC-surfaces and an integrable finite-dimensional Hamiltonian system. For a summary of their results, see [20]. There it is shown that to any solution $\phi = (\psi_1, \psi_2)^T$ of the first order equations (which resemble Dirac equations)

$$\partial\psi_1 = p\psi_2, \qquad \bar{\partial}\psi_2 = -p\psi_1, \qquad p = |\psi_1|^2 + |\psi_2|^2, \tag{3.1}$$

one can associate a CMC-surface immersed into $\mathbb{R}^3$ with radius vector $X(z, \bar{z})$ of the form (2.19)

$$\begin{aligned}
X_1 &= \int_\gamma \left(\psi_1^2 - \psi_2^2\right) dz' + \left(\bar{\psi}_1^2 - \bar{\psi}_2^2\right) d\bar{z}', \\
X_2 &= \int_\gamma \left(\psi_1^2 + \psi_2^2\right) dz' - \left(\bar{\psi}_1^2 + \bar{\psi}_2^2\right) d\bar{z}', \\
X_3 &= -\int_\gamma \psi_1\psi_2 dz' + \bar{\psi}_1\bar{\psi}_2 d\bar{z}',
\end{aligned} \tag{3.2}$$

where γ is an arbitrary curve, which does not depend on the trajectory but only on its endpoints z in $\mathbb{C}$. Note that these equations are really written for surfaces with $H = 1$.

To see how to reduce more general cases down to this case – see [19]. Note further that ψ_1 and ψ_2 are now functions of both z and $\bar{z}$. The formulae (3.1) are the starting point of our analysis of CMC-surfaces in this paper, and according to [4], we will refer to system (3.1) as the generalised Weierstrass (GW) system.

In this paper, we examine certain aspects of CMC-surfaces in $\mathbb{R}^n$ in the context of relating them to solutions of low dimensional sigma models. In particular, we focus our attention on constructing a Weierstrass representation for generic two-dimensional surfaces immersed in $\mathbb{R}^8$, whose explicit form has not been known up to now. For the sake of convenience our investigation starts with a derivation of the position vector $X(z,\bar{z})$ of a surface in $\mathbb{R}^3$ from the Lax pair for a GW system (3.1). As it was shown in [21] the GW system (3.1) is in a one-to-one correspondence with the solutions of the equations of the completely integrable two-dimensional Euclidean CP^1 sigma model

$$[\partial\bar{\partial}P, P] = 0, \tag{3.3}$$

where P is a projector

$$P = \frac{1}{A}\begin{pmatrix} 1 & \bar{w} \\ w & |w|^2 \end{pmatrix}, \qquad A = 1 + |w|^2, \tag{3.4}$$

or equivalently, the solutions of

$$\partial\bar{\partial}w - \frac{2\bar{w}}{1+|w|^2}\partial w \bar{\partial} w = 0. \tag{3.5}$$

In [22] it was shown that if ψ_1 and ψ_2 are solutions of the GW system (3.1), then function w defined by

$$w = \frac{\psi_1}{\bar{\psi}_2}, \tag{3.6}$$

is a solution of the equations of the CP^1 sigma model, namely, (3.5). The converse is also true [21]. Thus, if w is a solution of (3.5), then ψ_1 and ψ_2 of the GW system (3.1) have the form

$$\psi_1 = \epsilon w \frac{(\bar{\partial}\bar{w})^{1/2}}{1+|w|^2}, \qquad \psi_2 = \epsilon \frac{(\partial w)^{1/2}}{1+|w|^2}, \qquad p = \frac{|\partial w|}{1+|w|^2}, \qquad \epsilon = \pm 1. \tag{3.7}$$

Note that equation (2.8) with $H = 1$ for the unit normal vector $N = (n_1, n_2, n_3)$ to a CMC-surface adopts the well known form of the equation of the $SO(3)$ sigma model

$$\partial\bar{\partial}N + (\partial N, \bar{\partial}N)N = 0, \qquad (N,N) = 1. \tag{3.8}$$

Combining the map of the unit vector N onto the unit sphere S^2 with the stereographic projection, we obtain the Gauss map

$$w = \frac{n_1 + i n_2}{1 + n_3} = \frac{\psi_1}{\bar{\psi}_2}. \tag{3.9}$$

which satisfies the CP^1 model equation (3.5).

However, as shown by Zakharov and Mikhailov [23] the equation (3.3) can be considered as a compatibility condition for two linear spectral problems

$$\partial\Psi = \frac{2}{1+\lambda}[\partial P, P]\Psi, \qquad \bar{\partial}\Psi = \frac{2}{1-\lambda}[\bar{\partial}P, P], \tag{3.10}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter. The compatibility condition for (3.10) can also be written the form of a conservation law

$$\partial K - \bar{\partial}K^{\dagger} = 0, \tag{3.11}$$

where the traceless 2 by 2 matrices K and $K^{\dagger}$ expressed in terms of w have the form

$$\begin{aligned} K = [\bar{\partial}P, P] &= \frac{1}{A^2}\begin{pmatrix} \bar{w}\bar{\partial}w - w\bar{\partial}\bar{w} & \bar{\partial}\bar{w} + \bar{w}^2\partial w \\ -\bar{\partial}w - w^2\bar{\partial}\bar{w} & w\bar{\partial}\bar{w} - \bar{w}\bar{\partial}w \end{pmatrix} \\ -K^{\dagger} = [\partial P, P] &= \frac{1}{A^2}\begin{pmatrix} \bar{w}\partial w - w\partial\bar{w} & \partial\bar{w} + \bar{w}^2\partial w \\ -\partial w - w^2\partial\bar{w} & w\partial\bar{w} - \bar{w}\partial w \end{pmatrix}, \end{aligned} \tag{3.12}$$

and the Hermitian conjugate is denoted by †.

Next we derive the explicit form of matrices K and $K^{\dagger}$ in terms of ψ_1 and ψ_2 in order to find the corresponding conservation laws for the GW system (3.5). For computational purposes, it is useful to express the first derivatives of w in terms of ψ_1 and ψ_2. Note that in the CP^1 case the quantity J defined in (2.7) is given by

$$J = \frac{\partial W \partial \bar{W}}{A^2}.$$

Using (3.2) we find that it is given by

$$J = \bar{\psi}_1\partial\psi_2 - \psi_2\partial\bar{\psi}_1, \tag{3.13}$$

and so is a holomorphic function, i.e. which satisfies,

$$\bar{\partial}J = \bar{\partial}(\bar{\psi}_1\partial\psi_2 - \psi_2\partial\bar{\psi}_1) = -p\partial p + p\partial p = 0, \tag{3.14}$$

whenever (3.1) holds. Actually, in the CP^1 case J is a component of the energy-momentum tensor. Note also that (3.13) is different from (2.23). It becomes, formally, equal to it under the substitution $\bar{\psi}_1 \to \psi_1$. Using equations (3.1), (3.6) and (3.13) we can express the first derivatives of w in terms of ψ_1, ψ_2 and J

$$\partial w = A^2\psi_2^2, \qquad \bar{\partial}w = -\bar{J}\bar{\psi}_2^{-2}, \tag{3.15}$$

where

$$A = 1 + \frac{|\psi_1|^2}{|\psi_2|^2}. \tag{3.16}$$

As a consequence of (3.11) and (3.12) we find that the GW system possesses at least three further conservation laws

$$\begin{aligned} &\partial(\psi_1\bar{\psi}_2 + \bar{R}\bar{\psi}_1\psi_2) - \bar{\partial}(\bar{\psi}_1\psi_2 + R\psi_1\bar{\psi}_2) = 0, \\ &\partial\left(\psi_1^2 - \bar{R}\psi_2^2\right) + \bar{\partial}\left(\psi_2^2 - R\psi_1^2\right) = 0, \\ &\partial\left(\bar{\psi}_2^2 - \bar{R}\bar{\psi}_1^2\right) + \bar{\partial}\left(\bar{\psi}_1^2 - R\bar{\psi}_2^2\right) = 0, \end{aligned} \tag{3.17}$$

where we have introduced

$$R = \frac{J}{p^2}. \tag{3.18}$$

Note that formulae (3.17) differ from the conservation laws derived in [4] as they contain additional terms involving R. If we put $R = 0$ in equations (3.17) then we recover the expressions given in [4]

$$\partial(\psi_1\bar{\psi}_2) - \bar{\partial}(\bar{\psi}_1\psi_2) = 0, \qquad \partial\left(\psi_1^2\right) + \bar{\partial}\left(\psi_2^2\right) = 0, \qquad \partial\left(\bar{\psi}_2^2\right) + \bar{\partial}\left(\bar{\psi}_1^2\right) = 0. \tag{3.19}$$

As a result of the conservation laws (3.17), we can introduce three real-valued functions $X_i(z, \bar{z})$ given by

$$\begin{aligned}
X_1 &= \frac{i}{2}\int_\gamma \left[\bar{\psi}_1^2 + \psi_2^2 - R\left(\psi_1^2 + \bar{\psi}_2^2\right)\right] dz' - \left[\psi_1^2 + \bar{\psi}_2^2 - R\left(\bar{\psi}_1^2 + \psi_2^2\right)\right] d\bar{z}', \\
X_2 &= \frac{1}{2}\int_\gamma \left[\bar{\psi}_1^2 - \psi_2^2 + R\left(\psi_1^2 - \bar{\psi}_2^2\right)\right] dz' + \left[\psi_1^2 - \bar{\psi}_2^2 + R\left(\bar{\psi}_1^2 - \psi_2^2\right)\right] d\bar{z}', \\
X_3 &= -\int_\gamma \left[\bar{\psi}_1\psi_2 + R\psi_1\bar{\psi}_2\right] dz' + \left[\psi_1\bar{\psi}_2 + \bar{R}\bar{\psi}_1\psi_2\right] d\bar{z}',
\end{aligned} \tag{3.20}$$

where γ is any curve from a fixed point z in $\mathbb{C}$. The functions X_i, $i = 1, 2, 3$ can be considered as components of a position vector of a surface locally parametrised by z and $\bar{z}$ and immersed in $\mathbb{R}^3$

$$X(z, \bar{z}) = (X_1(z, \bar{z}), X_2(z, \bar{z}), X_3(z, \bar{z})). \tag{3.21}$$

Using conformal changes of coordinates on the surface $\mathbb{M}^2$ we can, without loss of generality, put $J = 1$ (when, of course $J \neq 0$). As a consequence it is easy to show that representation (3.20) with $R = 1/p^2$ cannot be reduced to the Weierstrass formulae (3.2). This means, as we will see latter, that the additional terms involving R play an important role in the construction of surfaces in $\mathbb{R}^3$.

The tangents and the normal unit vector to the surface $\mathbb{M}^2$ are given by

$$\begin{aligned}
\partial X &= \left(i\left[\bar{\psi}_1^2 + \psi_2^2 - R\left(\psi_1^2 + \bar{\psi}_2^2\right)\right], \left[\bar{\psi}_1^2 - \psi_2^2 + R\left(\psi_1^2 - \bar{\psi}_2^2\right)\right],\right. \\
&\qquad \left. -2\left(\bar{\psi}_1\psi_2 + R\psi_1\bar{\psi}_2\right)\right), \\
\bar{\partial} X &= \left(-i\left[\psi_1^2 + \bar{\psi}_2^2 - \bar{R}\left(\bar{\psi}_1^2 + \psi_2^2\right)\right], \left[\psi_1^2 - \bar{\psi}_2^2 + \bar{R}\left(\bar{\psi}_1^2 - \psi_2^2\right)\right],\right. \\
&\qquad \left. -2\left(\psi_1\bar{\psi}_2 + \bar{R}\bar{\psi}_1\psi_2\right)\right),
\end{aligned} \tag{3.22}$$

and

$$N = \frac{1}{p}\left(i\left(\bar{\psi}_1\bar{\psi}_2 - \psi_1\psi_2\right), \bar{\psi}_1\bar{\psi}_2 + \psi_1\psi_2, |\psi_1|^2 - |\psi_2|^2\right), \tag{3.23}$$

respectively. The first and second fundamental forms of the surface $\mathbb{M}^2$ are given by

$$\begin{aligned}
I &= (dX, dX) = 4\left(J dz^2 + p^2\left(1 + |R|^2\right) dz\, d\bar{z} + \bar{J} d\bar{z}^2\right), \\
II &= \left(d^2X, N\right) = \left(4J + R + \bar{R}\right) dz^2 + \left(2p + i(R - \bar{R})\right) dz d\bar{z} + \left(4\bar{J} - R - \bar{R}\right) d\bar{z}^2.
\end{aligned} \tag{3.24}$$

These quadratic forms contain the Hopf differential Jdz^2 and are invariant under any conformal changes of coordinates. The Gauss and mean curvatures are

$$K = -p^{-2}\partial\bar{\partial}\ln p, \qquad H = 1, \tag{3.25}$$

respectively.

Note that if $J = 0$ then $R = 0$ and so the components of the fundamental forms (3.24) become

$$g_{12} = 2p^2, \qquad g_{11} = g_{22} = 0, \qquad b_{12} = 2p, \qquad b_{11} = b_{22} = 0. \tag{3.26}$$

In this case the solutions of GW system (3.1) expressed in terms of w are represented by (3.7), where $w(z)$ is any holomorphic function. According to [24], the energy

$$E = \iint_D \frac{\partial w \bar{\partial} w}{1 + |w|^2}\, dz \wedge d\bar{z}, \tag{3.27}$$

is finite when the function $w(z)$ is a ratio of polynomials in z. Geometrically, such functions ψ_i parametrise an immersed sphere $S^2 \subset \mathbb{R}^3$, since $J = 0$ implies the proportionality of fundamental forms I and II.

Note also that if in equations (3.24) we put $J \neq 0$ then there is no conformal immersion of surfaces in $\mathbb{R}^3$. Hence, equations (3.22) and (3.23) imply that the representation (3.20) can admit different CMC-surfaces which globally have the same Gauss map (3.9). This is due to the fact that the tangent vectors ∂X and $\bar{\partial} X$ depend on J while the unit normal vector N is independent of J. In [25, 26], using the isometric immersions, formulae similar to (3.22) and (3.23) for particular cases of isothermic surfaces have been discussed.

Let us now discuss the meaning of conservation laws (3.17). As J is a holomorphic function so, according to (2.10), we are dealing with CMC-surfaces. If the CP^1 model is defined over S^2 then solutions w of (3.5) are either holomorphic or antiholomorphic functions and so $J = 0$. However, if the CP^1 model is defined on $\mathbb{R}^2$ then the function w is not necessarily holomorphic or antiholomorphic and $J \neq 0$. Note that when $J \neq 0$ the solutions are defined on $\mathbb{R}^2/\{a\}$, where $\{a\}$ is a small set of points of $\mathbb{R}^2$. Subtracting (3.19) from (3.17) and introducing new independent variables η and $\bar{\eta}$ according to

$$d\eta = J^{1/2}\, dz, \qquad d\bar{\eta} = \bar{J}^{1/2} d\bar{z}, \qquad \bar{\partial} J = 0, \tag{3.28}$$

we obtain the following set of expressions:

$$\begin{gathered} |J|^2 \left[\partial_\eta \left(\frac{\bar{\psi}_1 \psi_2}{p^2}\right) - \bar{\partial}_{\bar{\eta}} \left(\frac{\psi_1 \bar{\psi}_2}{p^2}\right)\right] = 0, \qquad |J|^2 \left[\partial_\eta \left(\frac{\psi_2^2}{p^2}\right) + \bar{\partial}_{\bar{\eta}} \left(\frac{\psi_1}{p^2}\right)\right] = 0, \\ |J|^2 \left[\partial_\eta \left(\frac{\bar{\psi}_1^2}{p^2}\right) + \bar{\partial}_{\bar{\eta}} \left(\frac{\bar{\psi}_2^2}{p^2}\right)\right] = 0, \end{gathered} \tag{3.29}$$

where the derivatives are abbreviated $\partial_\eta = \partial/\partial\eta$ and $\bar{\partial}\bar{\eta} = \partial/\partial\bar{\eta}$. Equations (3.29) suggest that we should consider two separate cases, namely $J = 0$ which has been already treated in [4] and $J \neq 0$. In the latter case, under the change of variables (3.28) the GW system (3.1) adopts the form

$$\partial_\eta \psi_1 = \frac{p}{J}\, \psi_2, \qquad \bar{\partial}_{\bar{\eta}} \psi_2 = -\frac{p}{\bar{J}}\, \psi_1, \tag{3.30}$$

and the expression for J, given in (3.13), provides the following differential constraint (DC) on ψ_1 and ψ_2

$$\bar{\psi}_1\partial_\eta\psi_2 - \psi_2\partial_\eta\bar{\psi}_1 = 1. \tag{3.31}$$

Keeping in mind that the complex coordinates η and $\bar{\eta}$ are is defined up to a conformal transformation, we can without loss of generality put $J = 1$. If $\psi_1 \neq 0$ then the system (3.30), subject to DC (3.31), can be written in an equivalent form

$$\partial\psi_1 = p\psi_2, \qquad \partial\psi_2 = \bar{\psi}^{-1}(1 + \psi_2\partial\bar{\psi}_1), \qquad \bar{\partial}\psi_2 = -p\psi_1. \tag{3.32}$$

The compatibility condition for (3.32) does not imply any new DC on first order derivatives of ψ_1. Hence, the system (3.32) is integrable and the derivatives $\bar{\partial}\psi_1$ and $\partial\psi_1$ are undetermined.

The Gaussian curvature and mean curvature are

$$K = \left(|\psi_2|^2 - |\psi_1|^2\right)\left[|\bar{\partial}\psi_1|^2 + |\psi_2|^{-2}\left(1 + \psi_2\partial\bar{\psi}_1 + \bar{\psi}_2\bar{\partial}\psi_1\right)\right], \qquad H = 1, \tag{3.33}$$

respectively.

Note that the equations of the complex frame (2.5) are specified by DC (3.31) and are compatible with the scalar products (2.3) and (2.4). After the change of dependent variables

$$p = e^{\varphi/2}, \tag{3.34}$$

the corresponding Gauss–Codazzi equations (2.9)–(2.10) take the form of the elliptic Sh-Gordon equation

$$\partial\bar{\partial}\varphi + 4\sinh\varphi = 0. \tag{3.35}$$

Hence the CMC-surfaces are determined by formulae (3.20), where ψ_1 and ψ_2 have to obey equations (3.32) with p determined by (3.34) and (3.35). In terms of arbitrary conformal coordinates, we have proved that (ψ_1, ψ_2, p) can be viewed as the Weierstrass data of the CMC-surface $\mathbb{M}^2$ in $\mathbb{R}^3$.

To summarize: the generalised Weierstrass representation for the immersion of a CMC-surface into $\mathbb{R}^3$ is described by formulae (3.20), where ψ_1 and ψ_2 obey the GW system of equations (3.1).

Let us add also, as shown in [27], that under the changes of independent variables (3.28) and dependent variables $S = 2p^2|J|^{-1}$, the GW system (3.1) is decoupled into a direct sum of equations

$$\partial_\eta\bar{\partial}_{\bar{\eta}}\ln S = S^{-1} - S, \qquad \bar{\partial}_{\bar{\eta}}J = 0, \qquad \partial_\eta\bar{J} = 0. \tag{3.36}$$

Hence, the GW system (3.1) is completely integrable. Examples of N solitons solutions can be found in [21].

4 The CP^2 maps and the Weierstrass representation for surfaces in eight dimensional Euclidean spaces

The aim of this section is to demonstrate a connection between the recently proposed generalised Weierstrass (GW) system [28]

$$\partial\psi_1 = \left(1 + \frac{|\psi_2|^2}{|\varphi_2|^2}\right)\bar{\varphi}_1\bar{Q} - \frac{1}{2}\left[\frac{\psi_1\bar{\psi}_2}{\varphi_2} + \frac{|\psi_1|^2}{|\varphi_1|^2}\frac{\bar{\varphi}_2^2}{\bar{\varphi}_1}\right]\bar{P}, \tag{4.1}$$

$$\partial\psi_2 = \left(1 + \frac{|\psi_1|^2}{|\varphi_1|^2}\right)\bar{\varphi}_2\bar{P} - \frac{1}{2}\left[\frac{\bar{\psi}_1\psi_2}{\varphi_1} + \frac{|\psi_2|^2}{|\varphi_2|^2}\frac{\bar{\varphi}_1^2}{\bar{\varphi}_2}\right]\bar{Q}, \tag{4.2}$$

$$\bar{\partial}\varphi_1 = -\frac{1}{2}\left[\left(\frac{\psi_2}{\bar{\varphi}_2}P + 2\frac{\psi_1}{\bar{\varphi}_1}Q\right)\varphi_1 + P\psi_1\frac{\varphi_2^2}{|\varphi_1|^2}\right], \tag{4.3}$$

$$\bar{\partial}\varphi_2 = -\frac{1}{2}\left[\left(\frac{\psi_1}{\bar{\varphi}_1}Q + 2\frac{\psi_2}{\bar{\varphi}_2}P\right)\varphi_2 + Q\psi_2\frac{\varphi_1^2}{|\varphi_2|^2}\right], \tag{4.4}$$

where

$$P = \frac{\psi_1\bar{\psi}_2\bar{\varphi}_1}{\varphi_2} + \left(|\varphi_2|^2 + |\psi_2|^2\right)\frac{\bar{\varphi}_2}{\varphi_2}, \qquad Q = \frac{\bar{\psi}_1\psi_2\bar{\varphi}_2}{\varphi_1} + \left(|\varphi_1|^2 + |\psi_1|^2\right)\frac{\bar{\varphi}_1}{\varphi_1} \tag{4.5}$$

and the equations of the CP^2 sigma model [24]

$$\partial\bar{\partial}w_1 - \frac{2\bar{w}_1}{A}\partial w_1\bar{\partial}w_1 - \frac{\bar{w}_2}{A}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0, \tag{4.6}$$

$$\partial\bar{\partial}w_2 - \frac{2\bar{w}_2}{A}\partial w_2\bar{\partial}w_2 - \frac{\bar{w}_1}{A}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0, \tag{4.7}$$

$$A = 1 + |w_1|^2 + |w_2|^2. \tag{4.8}$$

Next, we exploit this connection and use the conservation laws for the GW system (4.1)–(4.4) to define real valued functions $X^i(z,\bar{z})$, $i = 1,\ldots,8$ in terms of functions φ_α, ψ_α, $\alpha = 1,2$ which are identified as the coordinates in 8-dim Euclidean space $\mathbb{R}^8$. The formulae (4.1)–(4.4) and (4.6)–(4.8) are the starting point for our analysis. In this paper, when we refer to system (4.1)–(4.4), we will describe it as the modified version of the original Weierstrass system (3.1).

Note that in equations (4.1)–(4.4) only four out of eight derivatives of functions ψ_i and φ_i are known in terms of complex functions ψ_i and φ_i and their complex conjugates while the others are unspecified. Note also that if the functions ψ_α tend to $\psi/\sqrt{2}$ and φ_α tend to φ, i.e. then the system (4.1)–(4.4) reduces to the Weierstrass formulae (3.1) for the CMC-surfaces immersed in $\mathbb{R}^3$

$$\partial\psi = \left(|\psi|^2 + |\varphi|^2\right)\varphi, \qquad \bar{\partial}\varphi = -\left(|\psi|^2 + |\varphi|^2\right)\psi.$$

In terms of w_i, $i = 1,2$ the above limit takes the form

$$w_i \to \frac{1}{\sqrt{2}}w, \qquad i = 1,2, \tag{4.9}$$

and then the CP^2 sigma model (4.6)–(4.8) reduces to the CP^1 sigma model (3.5). These limits characterise the properties of the solutions of systems (4.1)–(4.4) and (4.6)–(4.8).

First we show that there exists a one to one correspondence between GW system (4.1)–(4.4) and the equations of the CP^2 sigma model (4.6)–(4.8). For this purpose, we define two new complex valued functions

$$w_1 = \frac{\psi_1}{\bar{\varphi}_1}, \qquad w_2 = \frac{\psi_2}{\bar{\varphi}_2} \tag{4.10}$$

and using the GW system (4.1)–(4.4), we obtain

$$\begin{aligned} \partial w_1 &= A\left[w_1 \bar{w}_2 \varphi_2^2 + \left(1 + |w_1|^2\right) \varphi_1^2\right], \\ \partial w_2 &= A\left[\bar{w}_1 w_2 \varphi_1^2 + \left(1 + |w_2|^2\right) \varphi_2^2\right]. \end{aligned} \tag{4.11}$$

These relations generate the following transformation from the variables (w_1, w_2) and their derivatives to the variables $(\varphi_1, \varphi_2, \psi_1, \psi_2)$

$$\varphi_1 = \epsilon A^{-1}\left[\left(1 + |w_2|^2\right) \partial w_1 - w_1 \bar{w}_2 \partial w_2\right]^{1/2}, \tag{4.12}$$

$$\varphi_2 = \epsilon A^{-1}\left[-\bar{w}_1 w_2 \partial w_1 + \left(1 + |w_1|^2\right) \partial w_2\right]^{1/2}, \tag{4.13}$$

$$\psi_1 = \epsilon w_1 A^{-1}\left[\left(1 + |w_2|^2\right) \bar{\partial} \bar{w}_1 - \bar{w}_1 w_2 \bar{\partial} \bar{w}_2\right]^{1/2}, \tag{4.14}$$

$$\psi_2 = \epsilon w_2 A^{-1}\left[-w_1 \bar{w}_2 \bar{\partial} \bar{w}_1 + \left(1 + |w_1|^2\right) \bar{\partial} \bar{w}_2\right]^{1/2}. \tag{4.15}$$

We can now state the following: if the complex valued functions $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ are solutions of GW system (4.1)–(4.4), then the functions (w_1, w_2), defined by (4.10), solve the equations of the CP^2 sigma model (4.6)–(4.8).

Conversely, if the complex valued functions (w_1, w_2) are solutions of the CP^2 sigma model equations (4.6)–(4.8), then the complex valued functions $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ defined by (4.12)–(4.15) in terms of functions (w_1, w_2) and their 1st derivatives satisfy the GW system (4.1)–(4.4).

The proof of our statement is straightforward. The differentiation of equations (4.11) with respect to z and $\bar{z}$, respectively, yields

$$\begin{aligned} \partial\bar{\partial} w_1 = A & \left[\bar{w}_2 \varphi_2^2 \bar{\partial} w_1 + w_1 \varphi_2^2 \bar{\partial} \bar{w}_2 + 2 w_1 \bar{w}_2 \varphi_2 \bar{\partial} \varphi_2 + 2\left(1 + |w_1|^2\right) \varphi_1 \bar{\partial} \varphi_1 \right. \\ & \left. + \left(\bar{w}_1 \bar{\partial} w_1 + w_1 \bar{\partial} \bar{w}_1\right) \varphi_1^2\right] + \left[w_1 \bar{w}_2 \varphi_2^2 + \left(1 + |w_1|^2\right) \varphi_1^2\right] \\ & \times \left(\bar{w}_1 \bar{\partial} w_1 + w_1 \bar{\partial} \bar{w}_1 + \bar{w}_2 \bar{\partial} w_2 + w_2 \bar{\partial} \bar{w}_2\right), \end{aligned}$$

and

$$\begin{aligned} \partial\bar{\partial} w_2 = A & \left[w_2 \varphi_1^2 \bar{\partial} \bar{w}_1 + \bar{w}_1 \varphi_1^2 \bar{\partial} w_2 + 2 \bar{w}_1 w_2 \varphi_1 \bar{\partial} \varphi_1 + 2\left(1 + |w_2|^2\right) \varphi_2 \bar{\partial} \varphi_2 \right. \\ & \left. + \left(\bar{w}_2 \bar{\partial} w_2 + w_2 \bar{\partial} \bar{w}_2\right) \varphi_2^2\right] + \left[\bar{w}_1 w_2 \varphi_1^2 + \left(1 + |w_2|^2\right) \varphi_2^2\right] \\ & \times \left(\bar{w}_1 \bar{\partial} w_1 + w_1 \bar{\partial} \bar{w}_1 + \bar{w}_2 \bar{\partial} w_2 + w_2 \bar{\partial} \bar{w}_2\right) \end{aligned}$$

and their respective complex conjugate equations.

Substituting (4.11) and into the left-hand side of the first equation (4.6), we obtain

$$\begin{aligned} & \partial\bar{\partial} w_1 - \frac{2\bar{w}_1}{A} \partial w_1 \bar{\partial} w_1 - \frac{\bar{w}_2}{A}\left(\partial w_1 \bar{\partial} w_2 + \bar{\partial} w_1 \partial w_2\right) \\ & \quad = \left[A w_1 \varphi_2^2 + w_1 |w_2|^2 \varphi_2^2 + \left(1 + |w_1|^2\right) w_2 \varphi_1^2\right]\left(w_1 \bar{w}_2 \bar{\varphi}_1^2 + \left(1 + |w_2|^2\right) \bar{\varphi}_2^2\right) \\ & \quad + \left[A w_1 \varphi_1^2 + w_1^2 \bar{w}_2^2 \varphi_2^2 + \left(1 + |w_1|^2\right) w_1 \varphi_1^2\right]\left(\bar{w}_1 w_2 \bar{\varphi}_2^2 + \left(1 + |w_1|^2\right) \bar{\varphi}_1^2\right) \\ & \quad + 2\left[w_1 \bar{w}_2 \varphi_2 \bar{\partial} \varphi_2 + \left(1 + |w_1|^2\right) \varphi_1 \bar{\partial} \varphi_1\right]. \end{aligned} \tag{4.16}$$

Making use of the equations (4.1)–(4.4) we find that the equation (4.16) is satisfied identically. An analogous result takes place for the second equation (4.7), since the CP^2 sigma model equations (4.6)–(4.8) are invariant under

$$w_1 \leftrightarrow w_2 \tag{4.17}$$

This observation implies that the left-hand side of (4.7) vanishes as well whenever (4.1)–(4.4) holds.

Conversely, differentiating (4.12) with respect to $\bar{z}$ and using (4.12), we get

$$\begin{aligned}\bar{\partial}\varphi_1 = \frac{1}{2A^2\varphi_1}&\left[\left(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2\right)\partial w_1 + \left(1+|w_2|^2\right)\partial\bar{\partial}w_1\right.\\&\left.- \left(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2\right)\partial w_2 - w_1\bar{w}_2\partial\bar{\partial}w_2\right]\\&- \frac{\varphi_1}{A}\left[\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1 + \bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2\right].\end{aligned} \tag{4.18}$$

Using equations (4.11) and (4.6)–(4.8), we can eliminate first and second derivatives of w_1 and w_2 in expression (4.18) and obtain

$$\begin{aligned}\bar{\partial}\varphi_1 = \frac{1}{2\varphi_1}&\left\{w_1^2\bar{w}_2|w_2|^2\bar{\varphi}_1^2\varphi_2^2 + \left(1+|w_2|^2\right)w_1|w_2|^2|\varphi_2|^4 + \left(1+|w_1|^2\right)w_1|w_2|^2|\varphi_1|^4\right.\\&+ \left(1+|w_1|^2\right)\left(1+|w_2|^2\right)w_2\varphi_1^2\bar{\varphi}_2^2 - w_1|w_1|^2|w_2|^2|\varphi_1|^4\\&- \left(1+|w_2|^2\right)|w_1|^2w_2\varphi_1^2\bar{\varphi}_2^2 - \left(1+|w_2|^2\right)w_1^2\bar{w}_2\bar{\varphi}_1^2\varphi_2 - \left(1+|w_2|^2\right)^2 w_1|\varphi_2|^4\\&- 2A\varphi_1^2\left(w_2\bar{\varphi}_2^2 + w_1\bar{\varphi}_1^2\right) + \frac{1}{A^2}\left(A\left[w_1\bar{w}_2^2\varphi_2^2 + \bar{w}_2\left(1+|w_1|^2\right)\varphi_1^2\right]\right.\\&+ \left(1+|w_2|^2\right)\left[w_1\bar{w}_2^2\varphi_2^2 + \bar{w}_2\left(1+|w_1|^2\right)\varphi_1^2\right]\\&\left.- \left(A+|w_2|^2\right)|w_1|^2\bar{w}_2\varphi_1^2 - \left(A+1+|w_2|^2\right)w_1\bar{w}_2^2\varphi_2^2 - 2A\bar{w}_2\varphi_1^2\right)\bar{\partial}w_2\\&+ \frac{1}{A^2}\left(\left(1+|w_2|^2\right)\left[\left(A+|w_1|^2\right)\bar{w}_2\varphi_2^2 + \left(A+1+|w_1|^2\right)\bar{w}_1\varphi_1^2\right]\right.\\&- A\left[\bar{w}_1|w_2|^2\varphi_1^2 + \left(1+|w_2|^2\right)\bar{w}_2\varphi_2^2\right]\\&\left.\left.- \left[\bar{w}_1|w_1|^2|w_2|^2\varphi_1^2 + \left(1+|w_2|^2\right)|w_1|^2\bar{w}_2\varphi_2^2\right] - 2A\bar{w}_1\varphi_1^2\right)\bar{\partial}w_1\right\}.\end{aligned} \tag{4.19}$$

Collecting all the coefficients of the derivatives $\bar{\partial}w_1$ and $\bar{\partial}w_2$ in expression (4.19) we find that these coefficients vanish identically. In fact, we have

$$\begin{aligned}\bar{\partial}w_1:\ &\left(A+1+|w_1|^2 + \left(A+1+|w_1|^2\right)|w_2|^2 - A|w_2|^2 - |w_1|^2|w_2|^2 - 2A\right)\bar{w}_1\varphi_1^2\\&+ \left(A+|w_1|^2 + \left(A+|w_1|^2\right)|w_2|^2 - A\left(1+|w_2|^2\right) - \left(1+|w_2|^2\right)|w_1|^2\right)\bar{w}_2\varphi_2^2 \equiv 0.\end{aligned}$$

and

$$\begin{aligned}\bar{\partial}w_2:\ &\left(A\left(1+|w_1|^2\right) + A + |w_1|^2|w_2|^2 - \left(A+|w_2|^2\right)|w_1|^2 - 2A\right)\bar{w}_2\varphi_1^2\\&+ \left(A+1+|w_2|^2 - A - 1 - |w_2|^2\right)w_1\bar{w}_2\varphi_2^2 \equiv 0.\end{aligned} \tag{4.20}$$

Hence, (4.20) becomes

$$\begin{aligned}\bar{\partial}\varphi_1 = -\frac{1}{2}&\left\{\bar{w}_2w_1^2\frac{\bar{\varphi}_1^2\varphi_2^2}{\varphi_1} + \left(1+|w_2|^2\right)w_1\frac{|\varphi_2|^4}{\varphi_1} + 2Aw_2\varphi_1\bar{\varphi}_2^2\right.\\&\left.+ 2Aw_1|\varphi_1|^2\varphi_1 - w_1|w_2|^2\frac{|\varphi_1|^4}{\varphi_1} - \left(1+|w_2|^2\right)w_2\varphi_1\bar{\varphi}_2\right\}\end{aligned}$$

Performing the transformation (4.10) we obtain the first equation of (4.1), i.e.

$$\bar{\partial}\varphi_1 = -\frac{1}{2}\Big\{\frac{\varphi_2}{\varphi_1}\bar{\psi}_2\psi_1^2 + \left(1+|w_2|^2\right)\frac{|\varphi_2|^4}{|\varphi_1|^2} + \left(A+1+|w_1|^2\right)|\varphi_1|^2\psi_1 + \left(A+|w_1|^2\right)\varphi_1\bar{\varphi}_2\psi_2\Big\}. \tag{4.21}$$

Since the equations (4.1)–(4.4) are invariant under

$$\varphi_1 \leftrightarrow \varphi_2, \tag{4.22}$$

an analogous result holds for (4.2). Differentiation of (4.10) with respect to z gives

$$\partial\psi_1 = \bar{\varphi}_1\partial w_1 + w_1\partial\bar{\varphi}_1. \tag{4.23}$$

Substituting (4.11) and the complex conjugate equation of (4.1) into (4.23) we get (4.3). Making use of the discrete symmetry (4.22) in (4.3), we obtain equation (4.4), which completes the proof.

An interesting property of the GW system (4.1)–(4.4) in the context of the CP^2 sigma model (4.6)–(4.8) is the existence of a gauge freedom in the definition of the variables w_1 and w_2 given by formula (4.10). This is due to the fact that the numerator and the denominator of (4.10) can be multiplied by any complex functions $f_i : \mathbb{C} \to \mathbb{C}$, $i = 1, 2$. This means that if we introduce a new set of complex valued functions $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ which are related to functions $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ in the following way

$$\varphi_i = f_i(z,\bar{z})\alpha_i, \qquad \psi_i = \bar{f}(z,\bar{z})\beta_i, \qquad i = 1, 2, \tag{4.24}$$

then the transformation (4.24) leaves the functions w_1, w_2 invariant

$$w_1 = \frac{\beta_1}{\bar{\alpha}_1}, \qquad w_2 = \frac{\beta_2}{\bar{\alpha}_2}. \tag{4.25}$$

We show now that if the complex valued functions w_1, w_2 are solutions of the CP^2 sigma model equations (4.6)–(4.8), then for any two holomorphic functions f_i, $i = 1, 2$ the complex functions $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ defined by

$$\begin{aligned}
&\alpha_1 = \epsilon f_1^{-1}A^{-1}\left[\left(1+|w_2|^2\right)\partial w_1 - w_1\bar{w}_2\partial w_2\right]^{1/2},\\
&\alpha_2 = \epsilon f_2^{-1}A^{-1}\left[-\bar{w}_1 w_2\partial w_1 + \left(1+|w_1|^2\right)\partial w_2\right]^{1/2},\\
&\beta_1 = \epsilon w_1\bar{f}_1^{-1}A^{-1}\left[\left(1+|w_2|^2\right)\bar{\partial}\bar{w}_1 - \bar{w}_1 w_2\bar{\partial}\bar{w}_2\right]^{1/2},\\
&\beta_2 = \epsilon w_2\bar{f}_2^{-1}A^{-1}\left[-w_1\bar{w}_2\bar{\partial}\bar{w}_1 + \left(1+|w_1|^2\right)\bar{\partial}\bar{w}_2\right]^{1/2}, \qquad \bar{\partial}f_i = 0,
\end{aligned} \tag{4.26}$$

satisfy the GW system (4.1)–(4.4).

Indeed, the result is obtained directly by substituting (4.24) and (4.25) into CP^2 sigma model equations (4.6)–(4.8). This leads to differential constraints for the functions f_i and their first derivatives

$$\left(f_i^2 - 1\right)\bar{\partial}f_j = 0, \qquad \left(\bar{f}_i^2 - 1\right)\partial\bar{f}_j = 0, \qquad i,j = 1, 2. \tag{4.27}$$

Hence, the general solutions of this system are given by any holomorphic functions f_i i.e.

$$\bar{\partial} f_i = 0, \qquad i = 1, 2 \tag{4.28}$$

Then invoking the main result of the last section we note that the transformation (4.12)–(4.15) becomes the one given by (4.26).

Another interesting property in the context of the CP^2 sigma model (4.6)–(4.8) and the GW system (4.1)–(4.4) is the existence of the quantity J [24] (which is a generalization of (3.17)).

$$J = A^{-2} \left\{ \partial w_1 \partial \bar{w}_1 + \partial w_2 \partial \bar{w}_2 + (\bar{w}_1 \partial \bar{w}_2 - \bar{w}_2 \partial \bar{w}_1)(w_1 \partial w_2 - w_2 \partial w_1) \right\}, \tag{4.29}$$

whose derivative with respect to z vanishes identically whenever equations (4.6)–(4.8) are satisfied

$$\bar{\partial} J = 0. \tag{4.30}$$

This means that J, given by (4.29), is holomorphic.

Note that if the functions $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ are solutions of GW system (4.1)–(4.4), then J, when written in terms of functions $(\varphi_1, \varphi_2, \psi_1, \psi_2)$, takes the form

$$J = \varphi_1 \partial \bar{\psi}_1 - \bar{\psi}_1 \partial \varphi_1 + \varphi_2 \partial \bar{\psi}_2 - \bar{\psi}_2 \partial \varphi_2 \tag{4.31}$$

and it satisfies

$$\bar{\partial} J = 0, \tag{4.32}$$

whenever equations (4.1)–(4.4) hold.

Next we exploit the observation [23] that the equations of the CP^2 sigma model (4.6)–(4.8) can be written as the compatibility condition for two linear spectral problems

$$\partial \Phi = \frac{2}{1+\lambda} [\partial P, P] \Phi, \qquad \bar{\partial} \Phi = \frac{2}{1-\lambda} [\bar{\partial} P, P] \Phi, \qquad \lambda \in \mathbb{C}, \tag{4.33}$$

where the 3×3 P is given by

$$P = A^{-1} M, \qquad M = \begin{pmatrix} 1 & w_1 & w_2 \\ \bar{w}_1 & |w_1|^2 & \bar{w}_1 w_2 \\ \bar{w}_2 & w_1 \bar{w}_2 & |w_2|^2 \end{pmatrix}, \tag{4.34}$$

and λ represents the spectral parameter. Using matrix P, the compatibility conditions of equations (4.33) imply

$$[\partial \bar{\partial} P, P] = 0, \tag{4.35}$$

which are satisfied whenever equations (4.6)–(4.8) hold. Equivalently, formula (4.35) can be rewritten, in a divergent form, as

$$\partial [\bar{\partial} P, P] + \bar{\partial} [\partial P, P] = 0. \tag{4.36}$$

Hence, from equations (4.34) and (4.36) we obtain the explicit form of the local conservation laws for the CP^2 sigma model

$$\partial K + \bar{\partial} L = 0, \tag{4.37}$$

where we have introduced the following notation for the traceless matrices K and L:

$$K = \frac{1}{A^2}\,[\bar{\partial} M, M], \qquad L = -K^\dagger = \frac{1}{A^2}\,[\partial M, M], \qquad \mathrm{tr}\, K = \mathrm{tr}\, L = 0. \tag{4.38}$$

Explicitly, the matrix elements of K and L are of the form

$$\begin{aligned}
k_{11} &= A^{-2}\left\{(\bar{w}_1\bar{\partial}w_1 + \bar{w}_2\bar{\partial}w_2) - (w_1\bar{\partial}\bar{w}_1 + w_2\bar{\partial}\bar{w}_2)\right\},\\
k_{12} &= A^{-2}\left\{|w_1|^2\bar{\partial}w_1 + w_1\bar{w}_2\bar{\partial}w_2 - (\bar{\partial}w_1 + w_1(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1)\right.\\
&\quad \left. + w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2))\right\},\\
k_{13} &= A^{-2}\left\{\bar{w}_1 w_2\bar{\partial}w_1 + |w_2|^2\bar{\partial}w_2 - (\bar{\partial}w_2 + w_1(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1)\right.\\
&\quad \left. + w_2(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2))\right\},\\
k_{21} &= A^{-2}\left\{\bar{\partial}\bar{w}_1 + \bar{w}_1(\bar{w}_1(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1) + \bar{w}_2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1)\right.\\
&\quad \left. - (|w_1|^2\bar{\partial}\bar{w}_1 + \bar{w}_1 w_2\bar{\partial}\bar{w}_2)\right\},\\
k_{22} &= A^{-2}\left\{w_1\bar{\partial}\bar{w}_1 + w_1\bar{w}_2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1) - (\bar{w}_1\bar{\partial}w_1 + \bar{w}_1 w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2))\right\},\\
k_{23} &= A^{-2}\left\{w_2\bar{\partial}\bar{w}_1 + \bar{w}_1 w_2(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1) + |w_2|^2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1)\right.\\
&\quad \left. - \left[\bar{w}_1\bar{\partial}w_2 + |w_1|^2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1) + \bar{w}_1 w_2(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2)\right]\right\},\\
k_{31} &= A^{-2}\left\{\bar{\partial}\bar{w}_2 + \bar{w}_1(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2) + \bar{w}_2(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2)\right.\\
&\quad \left. - \left(w_1\bar{w}_2\bar{\partial}\bar{w}_1 + |w_2|^2\bar{\partial}\bar{w}_2\right)\right\},\\
k_{32} &= A^{-2}\left\{w_1\bar{\partial}\bar{w}_2 + |w_1|^2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2) + w_1\bar{w}_2(\bar{w}_2\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_2)\right.\\
&\quad \left. - \left[\bar{w}_2\bar{\partial}w_1 + w_1\bar{w}_2(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1) + |w_2|^2|\bar{w}_2|^2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2)\right]\right\},\\
k_{33} &= A^{-2}\left\{w_2\bar{\partial}\bar{w}_2 + \bar{w}_1 w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2)\right.\\
&\quad \left. - (\bar{w}_2\bar{\partial}w_2 + w_1\bar{w}_2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1))\right\},
\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}
l_{11} &= A^{-2}\left\{\bar{w}_1\partial w_1 + \bar{w}_2\partial w_2 - (w_1\partial\bar{w}_1 + w_2\partial\bar{w}_2)\right\},\\
l_{12} &= A^{-2}\left\{|w_1|^2\partial w_1 + w_1\bar{w}_2\partial w_2 - [\partial w_1 + w_1(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1)\right.\\
&\quad \left. + w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)]\right\},\\
l_{13} &= A^{-2}\left\{\bar{w}_1 w_2\partial w_1 + |w_2|^2\partial w_2 - [\partial w_2 + w_1(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1)\right.\\
&\quad \left. + w_2(\bar{w}_2\partial w_2 + w_2\partial\bar{w}_2)]\right\},\\
l_{21} &= A^{-2}\left\{\partial\bar{w}_1 + \bar{w}_1(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1) + \bar{w}_2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1)\right.\\
&\quad \left. - (|w_1|^2\partial\bar{w}_1 + \bar{w}_1 w_2\partial\bar{w}_2)\right\},\\
l_{22} &= A^{-2}\left\{w_1\partial\bar{w}_1 + w_1\bar{w}_2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) - [\bar{w}_1\partial w_1 + \bar{w}_1 w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)]\right\},\\
l_{23} &= A^{-2}\left\{w_2\partial\bar{w}_1 + \bar{w}_1 w_2(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1) + |w_2|^2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1)\right.\\
&\quad \left. - [\bar{w}_1\partial w_2 + |w_1|^2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) + \bar{w}_1 w_2(\bar{w}_2\partial w_2 + w_2\partial\bar{w}_2)]\right\},
\end{aligned}$$

$$
\begin{aligned}
l_{31} &= A^{-2}\left\{\partial\bar{w}_2 + \bar{w}_1(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2) + \bar{w}_2(\bar{w}_2\partial w_2 + w_2\partial\bar{w}_2)\right. \\
&\quad \left. - (w_1\bar{w}_2\partial\bar{w}_1 + |w_2|^2\partial\bar{w}_2)\right\}, \\
l_{32} &= A^{-2}\left\{w_1\partial\bar{w}_2 + |w_1|^2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2) + w_1\bar{w}_2(\bar{w}_2\partial w_2 + w_2\partial\bar{w}_2)\right. \\
&\quad \left. - [\bar{w}_2\partial w_1 + w_1\bar{w}_2(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1) + |w_2|^2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)]\right\}, \\
l_{33} &= A^{-2}\left\{w_2\partial\bar{w}_2 + \bar{w}_1 w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)\right. \\
&\quad \left. - (\bar{w}_2\partial w_2 + w_1\bar{w}_2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1))\right\},
\end{aligned}
\tag{4.40}
$$

respectively. Finally from equations (4.37), (4.39) and (4.40) we see that there exists only five independent conservation laws for CP^2 sigma model (4.6)–(4.8). Namely we have the following independent conserved quantities

$$
\begin{aligned}
&\partial\left\{A^{-2}\left[\bar{w}_1\bar{\partial}w_1 + \bar{w}_2\bar{\partial}w_2 - (w_1\bar{\partial}\bar{w}_1 + w_2\bar{\partial}w_2)\right]\right\} \\
&\quad + \bar{\partial}\left\{A^{-2}\left[\bar{w}_1\partial w_1 + \bar{w}_2\partial w_2 - (w_1\partial\bar{w}_1 + w_2\partial\bar{w}_2)\right]\right\} = 0, \\
&\partial\left\{A^{-2}\left[w_1\bar{\partial}\bar{w}_1 + w_1\bar{w}_2(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1) - \bar{w}_1\bar{\partial}w_1 - \bar{w}_1 w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2)\right]\right\} \\
&\quad + \bar{\partial}\left\{A^{-2}\left[w_1\partial\bar{w}_1 + w_1\bar{w}_2(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) - \bar{w}_1\partial w_1\right.\right. \\
&\quad \left.\left. - \bar{w}_1 w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)\right]\right\} = 0, \\
&\partial\left\{A^{-2}\left[|w_1|^2\bar{\partial}w_1 + w_1\bar{w}_2\bar{\partial}w_2 - \bar{\partial}w_1 - w_1(\bar{w}_1\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_1)\right.\right. \\
&\quad \left.\left. - w_2(\bar{w}_2\bar{\partial}w_1 + w_1\bar{\partial}\bar{w}_2)\right]\right\} + \bar{\partial}\left\{A^{-2}\left[|w_1|^2\partial w_1 + w_1\bar{w}_2\partial w_2 - \partial w_1\right.\right. \\
&\quad \left.\left. - w_1(\bar{w}_1\partial w_1 + w_1\partial\bar{w}_1) - w_2(\bar{w}_2\partial w_1 + w_1\partial\bar{w}_2)\right]\right\} = 0, \\
&\partial\left\{A^{-2}\left[\bar{w}_1 w_2\bar{\partial}w_1 - \bar{\partial}w_2 - w_1(\bar{w}_1\bar{\partial}w_2 + w_2\bar{\partial}\bar{w}_1) - w_2^2\bar{\partial}\bar{w}_2\right]\right\} \\
&\quad + \bar{\partial}\left\{A^{-2}\left[\bar{w}_1 w_2\partial w_1 - \partial w_2 - w_1(\bar{w}_1\partial w_2 + w_2\partial\bar{w}_1) - w_2^2\partial\bar{w}_2\right]\right\} = 0, \\
&\partial\left\{A^{-2}\left[w_2\bar{\partial}\bar{w}_1 + \bar{w}_1^2 w_2\bar{\partial}w_1 + |w_2|^2 w_2\bar{\partial}\bar{w}_1\right]\right. \\
&\quad \left. - A^{-2}\left[\bar{w}_1\bar{\partial}w_2 + |w_1|^2\bar{w}_1\bar{\partial}w_2 + \bar{w}_1 w_2^2\bar{\partial}\bar{w}_2\right]\right\} \\
&\quad + \bar{\partial}\left\{A^{-2}\left[w_2\partial\bar{w}_1 + \bar{w}_1^2 w_2\partial w_1 + |w_2|^2 w_2\partial\bar{w}_1\right]\right. \\
&\quad \left. - A^{-2}\left[\bar{w}_1\partial w_2 + |w_1|^2\bar{w}_1\partial w_2 + \bar{w}_1 w_2^2\partial\bar{w}_2\right]\right\} = 0.
\end{aligned}
\tag{4.41}
$$

Consequently, as a result of the conservation laws (4.41) there exist eight real-valued functions $X^i(z,\bar{z})$, $i = 1,\ldots,8$ expressed in terms of functions (w_1, w_2), i.e.

$$
\begin{aligned}
X^1 &= \int_C A^{-2}\left\{-\left[\bar{w}_1\partial w_1 + \bar{w}_2\partial w_2 - (w_1\partial\bar{w}_1 + w_2\partial\bar{w}_2)\right]dz\right. \\
&\quad \left. + \left[\bar{w}_1\bar{\partial}w_1 + \bar{w}_2\bar{\partial}w_2 - (w_1\bar{\partial}\bar{w}_1 + w_2\bar{\partial}\bar{w}_2)\right]d\bar{z}\right\}, \\
X^2 &= \int_C A^{-2}\left\{\left[\left(1+|w_2|^2\right)(w_1\partial\bar{w}_1 - \bar{w}_1\partial w_1) + |w_1|^2(\bar{w}_2\partial w_2 - w_2\partial\bar{w}_2)\right]dz\right. \\
&\quad \left. + \left[\left(1+|w_2|^2\right)(w_1\bar{\partial}\bar{w}_1 - \bar{w}_1\bar{\partial}w_1) + |w_1|^2(\bar{w}_2\bar{\partial}w_2 - w_2\bar{\partial}\bar{w}_2)\right]d\bar{z}\right\}, \\
X^3 &= i\int_C -A^{-2}\left\{\left[-\left(1+\bar{w}_1^2+|w_2|^2\right)\partial w_1\right.\right. \\
&\quad \left. - \left(1+w_1^2+|w_2|^2\right)\partial\bar{w}_1 + \bar{w}_2(w_1-\bar{w}_1)\partial w_2 + w_2(\bar{w}_1-w_1)\partial\bar{w}_2\right]dz \\
&\quad + \left[-\left(1+\bar{w}_1^2+|w_2|^2\right)\bar{\partial}w_1 - \left(1+w_1^2+|w_2|^2\right)\bar{\partial}\bar{w}_1 + \bar{w}_2(w_1-\bar{w}_1)\bar{\partial}w_2\right. \\
&\quad \left.\left. + w_2(\bar{w}_1-w_1)\bar{\partial}\bar{w}_2\right]d\bar{z}\right\},
\end{aligned}
$$

$$
\begin{aligned}
X^4 = &\int_C A^{-2}\left\{\left[\left(1-\bar{w}_1^2+|w_2|^2\right)\partial w_1\right.\right.\\
&+\left(-1+w_1^2-|w_2|^2\right)\partial\bar{w}_1-\bar{w}_2(w_1+\bar{w}_1)\partial w_2+w_2(w_1+\bar{w}_1)\partial\bar{w}_2\big]\,dz\\
&+\left[\left(-1+\bar{w}_1^2-|w_2|^2\right)\bar\partial w_1+\left(1-w_1^2+|w_2|^2\right)\bar\partial\bar{w}_1+\bar{w}_2(w_1+\bar{w}_1)\bar\partial w_2\right.\\
&\left.\left.-\,w_2(w_1+\bar{w}_1)\bar\partial\bar{w}_2\right]d\bar{z}\right\},\\
X^5 = i&\int_C -A^{-2}\left\{[\bar{w}_1(w_2-\bar{w}_2)\partial w_1+w_1(\bar{w}_2-w_2)\partial\bar{w}_1\right.\\
&-\left(1+|w_1|^2+\bar{w}_2^2\right)\partial w_2-\left(1+|w_1|^2+w_2^2\right)\partial\bar{w}_2\big]\,dz\\
&+\left[\bar{w}_1(w_2-\bar{w}_2)\bar\partial w_1+w_1(\bar{w}_2-w_2)\bar\partial\bar{w}_1-\left(1+|w_1|^2+\bar{w}_2^2\right)\bar\partial w_2\right.\\
&\left.\left.-\left(1+|w_1|^2+w_2^2\right)\bar\partial\bar{w}_2\right]d\bar{z}\right\},\\
X^6 = &\int_C A^{-2}\left\{[-\bar{w}_1(w_2+\bar{w}_2)\partial w_1+w_1(w_2+\bar{w}_2)\partial\bar{w}_1\right.\\
&+\left(1+|w_1|^2-\bar{w}_2^2\right)\partial w_2-\left(1+|w_1|^2-w_2^2\right)\partial\bar{w}_2\big]\,dz\\
&+\left[\bar{w}_1(w_2+\bar{w}_2)\bar\partial w_1-w_1(w_2+\bar{w}_2)\bar\partial\bar{w}_2-\left(1+|w_1|^2-\bar{w}_2^2\right)\bar\partial w_2\right.\\
&\left.\left.+\left(1+|w_1|^2-w_2^2\right)\bar\partial\bar{w}_2\right]d\bar{z}\right\},\\
X^7 = i&\int_C -A^{-2}\left\{\left[\bar{w}_2\left(1+|w_2|^2\right)+\bar{w}_1^2w_2\right]\partial w_1+\left[w_2(1+|w_2|^2)+w_1^2\bar{w}_2\right]\partial\bar{w}_1\right.\\
&\left.-\left[\bar{w}_1\left(1+|w_1|^2\right)+w_1\bar{w}_2^2\right]\partial w_2-\left[w_1\left(1+|w_1|^2\right)+\bar{w}_1w_2^2\right]\partial\bar{w}_2\right\}dz\\
&+A^{-2}\left\{\left[\bar{w}_2\left(1+|w_2|^2\right)+\bar{w}_1^2w_2\right]\bar\partial w_1+\left[w_2\left(1+|w_2|^2\right)+w_1^2\bar{w}_2\right]\bar\partial\bar{w}_1\right.\\
&\left.-\left[\bar{w}_1\left(1+|w_1|^2\right)+w_1\bar{w}_2^2\right]\bar\partial w_2-\left[w_1\left(1+|w_1|^2\right)+\bar{w}_1w_2^2\right]\bar\partial\bar{w}_2\right\}d\bar{z},\\
X^8 = &\int_C A^{-2}\left\{\left[\bar{w}_2\left(1+|w_2|^2\right)-\bar{w}_1^2w_2\right]\partial w_1+\left[-w_2\left(1+|w_2|^2\right)+w_1^2\bar{w}_2\right]\partial\bar{w}_1\right.\\
&\left.+\left[\bar{w}_1\left(1+|w_1|^2\right)-w_1\bar{w}_2^2\right]\partial w_2+\left[-w_1\left(1+|w_1|^2\right)+\bar{w}_1w_2^2\right]\partial\bar{w}_2\right\}dz\\
&+A^{-2}\left\{\left[-\bar{w}_2\left(1+|w_2|^2\right)+\bar{w}_1^2w_2\right]\bar\partial w_1+\left[w_2\left(1+|w_2|^2\right)-w_1^2\bar{w}_2^2\right]\bar\partial\bar{w}_1\right.\\
&\left.+\left[-\bar{w}_1\left(1+|w_1|^2\right)+w_1\bar{w}_2^2\right]\bar\partial w_2+\left[w_1\left(1+|w_1|^2\right)-\bar{w}_1w_2^2\right]\bar\partial\bar{w}_2\right\}d\bar{z}. \qquad (4.42)
\end{aligned}
$$

Note that by virtue of the conservation laws (4.41) for the CP^2 sigma model (4.6)–(4.8) the r.h.s. in expression (4.42) do not depend on the choice of the contour C but only on its endpoints. This is due to the fact that (4.39) are integrals of exact differentials of real valued functions. We identify the functions $X^i(z,\bar{z})$, $i=1,\dots,8$ with the coordinates of the radius vector

$$
\boldsymbol{X}(z,\bar{z})=\left(X^1(z,\bar{z}),\dots,X^8(z,\bar{z})\right), \qquad (4.43)
$$

of a two-dimensional surface immersed into eight-dimensional Euclidean space $\mathbb{R}^8$. Substituting (4.10) into (4.42) we express the radius vector $\boldsymbol{X}(z,\bar{z})$ in terms of functions $(\varphi_1,\varphi_2,\psi_1,\psi_2)$ and obtain

$$
\begin{aligned}
X^1 &= 2\int_C(\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)dz+(\psi_1\bar\varphi_1+\psi_2\bar\varphi_2)d\bar{z},\\
X^2 &= 2\int_C\left\{\frac{\bar\psi_1}{\varphi_1}\left[\varphi_1^2-\frac{\psi_1}{\bar\varphi_1}(\bar\psi_1\varphi_1+\bar\psi_2\varphi_2)-\frac{\psi_1}{\bar\varphi_1}\partial\left(A^{-1}\right)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{A\bar{\varphi}_2\psi_1}{\Omega}\left[\frac{J}{A^2}\frac{\psi_2}{\bar{\varphi}_2} + \left(\partial\left(A^{-1}\right) + \bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2\right)\varphi_2^2\right]\Bigg\}\, dz \\
&+ \Bigg\{\frac{\psi_1}{\bar{\varphi}_1}\left[\bar{\varphi}_1^2 - \frac{\bar{\psi}_1}{\varphi_1}(\psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2) - \frac{\bar{\psi}_1}{\varphi_1}\bar{\partial}\left(A^{-1}\right)\right] \\
&- \frac{A\varphi_2\bar{\psi}_1}{\bar{\Omega}}\left[\frac{\bar{J}}{A^2}\frac{\bar{\psi}_2}{\varphi_2} + \left(\bar{\partial}\left(A^{-1}\right) + \psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2\right)\bar{\varphi}_2^2\right]\Bigg\}\, d\bar{z},
\end{aligned}
$$

$$
\begin{aligned}
X^3 = i\int_C -&\Bigg\{-\frac{\bar{\psi}_1}{\varphi_1}\left[\partial\left(A^{-1}\right) + 2(\bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2)\right] \\
&- \frac{A\bar{\varphi}_1\bar{\varphi}_2}{\Omega}\left[\frac{J}{A^2}\frac{\psi_2}{\bar{\varphi}_2} + \left(\partial\left(A^{-1}\right) + \bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2\right)\varphi_2^2\right] \\
&+ \left[-\varphi_1^2 + \frac{\psi_1}{\bar{\varphi}_1}(\bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2) + \frac{\psi_1}{\bar{\varphi}_1}\partial\left(A^{-1}\right)\right]\Bigg\}\, dz \\
&+ \Bigg\{-\frac{\psi_1}{\bar{\varphi}_1}\left[\bar{\partial}\left(A^{-1}\right) + 2(\psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2)\right] \\
&- \frac{A\varphi_1\varphi_2}{\bar{\Omega}}\left[\frac{\bar{J}}{A^2}\frac{\bar{\psi}_2}{\varphi_2} + \left(\bar{\partial}\left(A^{-1}\right) + \psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2\right)\bar{\varphi}_2^2\right] \\
&+ \left[-\bar{\varphi}_1^2 + \frac{\bar{\psi}_1}{\varphi_1}(\psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2) + \frac{\bar{\psi}_1}{\varphi_1}\bar{\partial}\left(A^{-1}\right)\right]\Bigg\}\, d\bar{z},
\end{aligned}
$$

$$
\begin{aligned}
X^4 = \int_C &\Bigg\{-\frac{\bar{\psi}_1}{\varphi_1}\left[\partial\left(A^{-1}\right) + 2(\bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2)\right] \\
&- \frac{A\bar{\varphi}_1\bar{\varphi}_2}{\Omega}\left[\frac{J}{A^2}\frac{\psi_2}{\bar{\varphi}_2} + \left(\partial\left(A^{-1}\right) + \bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2\right)\varphi_2^2\right] \\
&- \left[-\varphi_1^2 + \frac{\psi_1}{\bar{\varphi}_1}(\bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2) + \frac{\psi_1}{\bar{\varphi}_1}\partial\left(A^{-1}\right)\right]\Bigg\}\, dz \\
&+ \Bigg\{-\frac{\psi_1}{\bar{\varphi}_1}\left[\bar{\partial}\left(A^{-1}\right) + 2(\psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2)\right] \\
&- \frac{A\varphi_1\varphi_2}{\bar{\Omega}}\left[\frac{\bar{J}}{A^2}\frac{\bar{\psi}_2}{\varphi_2} + \left(\bar{\partial}\left(A^{-1}\right) + \psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2\right)\bar{\varphi}_2^2\right] \\
&- \left[-\bar{\varphi}_1^2 + \frac{\bar{\psi}_1}{\varphi_1}(\psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2) + \frac{\bar{\psi}_1}{\varphi_1}\bar{\partial}\left(A^{-1}\right)\right]\Bigg\}\, d\bar{z},
\end{aligned}
$$

$$
\begin{aligned}
X^5 = i\int_C -&\Bigg\{-\frac{\bar{\psi}_2}{\varphi_2}\left[\partial\left(A^{-1}\right) + 2(\bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2)\right] \\
&+ \frac{A\bar{\varphi}_1\bar{\varphi}_2}{\Omega}\left[\left(\partial\left(A^{-1}\right) + \bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2\right)\varphi_1^2 + \frac{J}{A^2}\frac{\psi_1}{\bar{\varphi}_1}\right] \\
&+ \left[-\varphi_2^2 + \frac{\psi_2}{\bar{\varphi}_2}(\bar{\psi}_1\varphi_1 + \bar{\psi}_2\varphi_2) + \frac{\psi_2}{\bar{\varphi}_2}\partial\left(A^{-1}\right)\right]\Bigg\}\, dz \\
&+ \Bigg\{-\frac{\psi_2}{\bar{\varphi}_2}\left[\bar{\partial}\left(A^{-1}\right) + 2(\psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2)\right] \\
&+ \frac{A\varphi_1\varphi_2}{\bar{\Omega}}\left[\left(\bar{\partial}\left(A^{-1}\right) + \psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2\right)\bar{\varphi}_1^2 + \frac{\bar{J}}{A^2}\frac{\bar{\psi}_1}{\varphi_1}\right] \\
&+ \left[-\bar{\varphi}_2^2 + \frac{\bar{\psi}_2}{\varphi_2}(\psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2) + \frac{\bar{\psi}_2}{\varphi_2}\bar{\partial}\left(A^{-1}\right)\right]\Bigg\}\, d\bar{z},
\end{aligned}
$$

$$
\begin{aligned}
X^6 = \int_C \Bigg\{ & -\frac{\bar\psi_2}{\varphi_2}\left[\partial\left(A^{-1}\right) + 2(\bar\psi_1\varphi_1 + \bar\psi_2\varphi_2)\right] \\
& + \frac{A\bar\varphi_1\bar\varphi_2}{\Omega}\left[\left(\partial\left(A^{-1}\right) + \bar\psi_1\varphi_1 + \bar\psi_2\varphi_2\right)\varphi_1^2 + \frac{J}{A^2}\frac{\psi_1}{\bar\varphi_1}\right] \\
& - \left[-\varphi_2^2 + \frac{\psi_2}{\bar\varphi_2}(\bar\psi_1\varphi_1 + \bar\psi_2\varphi_2) + \frac{\psi_2}{\bar\varphi_2}\partial\left(A^{-1}\right)\right]\Bigg\}\, dz \\
& + \Bigg\{ -\frac{\psi_2}{\bar\varphi_2}\left[\bar\partial\left(A^{-1}\right) + 2(\psi_1\bar\varphi_1 + \psi_2\bar\varphi_2)\right] \\
& + \frac{A\varphi_1\varphi_2}{\bar\Omega}\left[\left(\bar\partial\left(A^{-1}\right) + \psi_1\bar\varphi_1 + \psi_2\bar\varphi_2\right)\bar\varphi_1^2 + \frac{\bar J}{A^2}\frac{\bar\psi_1}{\varphi_1}\right] \\
& - \left[-\bar\varphi_2^2 + \frac{\bar\psi_2}{\varphi_2}(\psi_1\bar\varphi_1 + \psi_2\bar\varphi_2) + \frac{\bar\psi_2}{\varphi_2}\bar\partial\left(A^{-1}\right)\right]\Bigg\}\, d\bar z, \\
X^7 = i\int_C - \Bigg\{ & \frac{\bar\psi_2}{\varphi_2}\left[\varphi_1^2 - \frac{\psi_1}{\bar\varphi_1}(\bar\psi_1\varphi_1 + \bar\psi_2\varphi_2) - \frac{\psi_1}{\bar\varphi_1}\partial\left(A^{-1}\right)\right] \\
& + \frac{A\bar\varphi_2\psi_1}{\Omega}\left[\left(\partial\left(A^{-1}\right) + \bar\psi_1\varphi_1 + \bar\psi_2\varphi_2\right)\varphi_1^2 + \frac{J}{A^2}\frac{\psi_1}{\bar\varphi_1}\right] \\
& + \frac{A\bar\varphi_1\psi_2}{\Omega}\left[\frac{J}{A^2}\frac{\psi_2}{\bar\varphi_2} + \left(\partial\left(A^{-1}\right) + \bar\psi_1\varphi_1 + \bar\psi_2\varphi_2\right)\varphi_2^2\right] \\
& + \frac{\bar\psi_1}{\varphi_1}\left[-\varphi_2^2 + \frac{\psi_2}{\bar\varphi_2}(\bar\psi_1\varphi_1 + \bar\psi_2\varphi_2) + \frac{\psi_2}{\bar\varphi_2}\partial\left(A^{-1}\right)\right]\Bigg\}\, dz \\
& + \Bigg\{ \frac{\psi_2}{\bar\varphi_2}\left[\bar\varphi_1^2 - \frac{\bar\psi_1}{\varphi_1}(\psi_1\bar\varphi_1 + \psi_2\bar\varphi_2) - \frac{\bar\psi_1}{\varphi_1}\bar\partial\left(A^{-1}\right)\right] \\
& + \frac{A\varphi_2\bar\psi_1}{\Omega}\left[\left(\bar\partial\left(A^{-1}\right) + \psi_1\bar\varphi_1 + \psi_2\bar\varphi_2\right)\bar\varphi_1^2 + \frac{\bar J}{A^2}\frac{\bar\psi_1}{\varphi_1}\right] \\
& + \frac{A\varphi_1\bar\psi_2}{\bar\Omega}\left[\frac{\bar J}{A^2}\frac{\bar\psi_2}{\varphi_2} + \left(\bar\partial\left(A^{-1}\right) + \psi_1\bar\varphi_1 + \psi_2\bar\varphi_2\right)\bar\varphi_2^2\right] \\
& + \frac{\psi_1}{\bar\varphi_1}\left[-\bar\varphi_2^2 + \frac{\bar\psi_2}{\varphi_2}(\psi_1\bar\varphi_1 + \psi_2\bar\varphi_2) + \frac{\bar\psi_2}{\varphi_2}\bar\partial\left(A^{-1}\right)\right]\Bigg\}\, d\bar z, \\
X^8 = \int_C \Bigg\{ & \frac{\bar\psi_2}{\varphi_2}\left[\varphi_1^2 - \frac{\psi_1}{\bar\varphi_1}(\bar\psi_1\varphi_1 + \bar\psi_2\varphi_2) - \frac{\psi_1}{\bar\varphi_1}\partial\left(A^{-1}\right)\right] \\
& + \frac{A\bar\varphi_2\psi_1}{\Omega}\left[\left(\partial\left(A^{-1}\right) + \bar\psi_1\varphi_1 + \bar\psi_2\varphi_2\right)\varphi_1^2 + \frac{J}{A^2}\frac{\psi_1}{\bar\varphi_1}\right] \\
& - \frac{A\bar\varphi_1\psi_2}{\Omega}\left[\frac{J}{A^2}\frac{\psi_2}{\bar\varphi_2} + \left(\partial\left(A^{-1}\right) + \bar\psi_1\varphi_1 + \bar\psi_2\varphi_2\right)\varphi_2^2\right] \\
& - \frac{\bar\psi_1}{\varphi_1}\left[-\varphi_2^2 + \frac{\psi_2}{\bar\varphi_2}(\bar\psi_1\varphi_1 + \bar\psi_2\varphi_2) + \frac{\psi_2}{\bar\varphi_2}\partial\left(A^{-1}\right)\right]\Bigg\}\, dz \\
& + \Bigg\{ \frac{\psi_2}{\bar\varphi_2}\left[\bar\varphi_1^2 - \frac{\bar\psi_1}{\varphi_1}(\psi_1\bar\varphi_1 + \psi_2\bar\varphi_2) - \frac{\bar\psi_1}{\varphi_1}\bar\partial\left(A^{-1}\right)\right] \\
& + \frac{A\varphi_2\bar\psi_1}{\bar\Omega}\left[\left(\bar\partial\left(A^{-1}\right) + \psi_1\bar\varphi_1 + \psi_2\bar\varphi_2\right)\bar\varphi_1^2 + \frac{\bar J}{A^2}\frac{\bar\psi_1}{\varphi_1}\right] \\
& - \frac{A\varphi_1\bar\psi_2}{\bar\Omega}\left[\frac{\bar J}{A^2}\frac{\bar\psi_2}{\varphi_2} + \left(\bar\partial\left(A^{-1}\right) + \psi_1\bar\varphi_1 + \psi_2\bar\varphi_2\right)\bar\varphi_2^2\right]
\end{aligned}
$$

$$- \frac{\psi_1}{\bar{\varphi}_1}\left[-\bar{\varphi}_2^2 + \frac{\bar{\psi}_2}{\varphi_2}(\psi_1\bar{\varphi}_1 + \psi_2\bar{\varphi}_2) + \frac{\bar{\psi}_2}{\varphi_2}\bar{\partial}\left(A^{-1}\right)\right]\Bigg\} d\bar{z}, \tag{4.44}$$

where we have introduced the following notation

$$\Omega = \varphi_1\psi_2|\varphi_1|^2 - \varphi_2\psi_1|\varphi_2|^2, \qquad \bar{\Omega} = \bar{\varphi}_1\bar{\psi}_2|\varphi_1|^2 - \bar{\varphi}_2\bar{\psi}_1|\varphi_2|^2. \tag{4.45}$$

Note that when J vanishes, as can be checked, the position vector X given by (4.44) obeys the following relations

$$(\partial X, \partial X) = 0, \tag{4.46}$$

and

$$(\partial\bar{\partial}X, \partial\bar{\partial}X) = (\partial X, \bar{\partial}X)^2 \neq 0, \tag{4.47}$$

whenever the equations of the CP^2 sigma model (4.6)–(4.8) are satisfied. The explicit form of (4.47), when written in terms of w's or ψ's and ϕ's, is very complicated and so we shall not reproduce it here. As a consequence of (4.46) and (4.47) the components of induced metric are

$$g_{zz} = g_{\bar{z}\bar{z}} = 0, \qquad g_{z\bar{z}} \neq 0 \tag{4.48}$$

and the norm of the mean curvature vector $\bar{H} = (g_{z\bar{z}})^{-1}\partial\bar{\partial}X$, as expected, is equal to one, i.e. $|\bar{H}|^2 = 1$. We would like to note that when w_i are holomorphic functions then $J = 0$ and formulae (4.42) define a surface on $SU(3)$. Then using expressions in [29] we can calculate, in a closed form, all geometric characteristics of a given surface.

Thus we have proved that the conformal immersions of CMC-surfaces into $\mathbb{R}^8$ are determined by formulae (4.42) or (4.44), where the complex functions w_i obey the equations of CP^2 sigma model (4.6)–(4.8), (or complex functions ψ_i and ϕ_i obey the first order system (4.1)–(4.4)) and J given by (4.29) (or (4.31)) vanishes.

5 Examples and applications

In this section, based on our results of previous sections we construct certain classes of two-dimensional CMC-surfaces immersed into $\mathbb{R}^8$. For this purpose we use the CP^2 sigma model defined over S^2. Note that for such a model all solutions of the Euler–Lagrange equations (4.6)–(4.8) are well known [24]. Under the requirement of the finiteness of the action they split into three separate classes, i.e. analytic (i.e. $w_i = w_i(z)$), antianalytic (i.e. $w_i = w_i(\bar{z})$) and mixed ones. The latter ones can be determined from either the holomorphic or antiholomorphic functions by the following procedure.

Consider three arbitrary holomorphic functions $f_i = f_i(z)$ and define for each pair the Wronskian

$$F_{ij} = f_i\partial f_j - f_j\partial f_i, \qquad \bar{\partial}f_i = 0, \qquad i,j = 1,2,3 \tag{5.1}$$

Next determine three complex valued functions

$$g_i = \sum_{k\neq i}^{3} \bar{f}_k(f_k\partial f_i - f_i\partial f_k), \qquad i = 1,2,3 \tag{5.2}$$

Then the mixed solutions w_i of CP^2 sigma model (4.6)–(4.8) can be determined as ratios of the components of g_i, i.e.

$$w_1 = \frac{g_1}{g_3}, \qquad w_2 = \frac{g_2}{g_3}, \qquad g_3 \neq 0. \tag{5.3}$$

Alternatively, similar class of solutions can be obtained when we consider three arbitrary antiholomorphic functions $f_i = f_i(\bar{z})$ and construct g_i in the same way as above, but using $\bar{\partial}$ instead of ∂ in the equations (5.2).

Now, let us discuss some classes of CMC-surfaces in $\mathbb{R}^n$ which can be obtained directly by applying the Weierstrass representation (4.42) and (5.3).

1. One of the simplest solutions which corresponds to the analytic choice of functions is

$$w_1 = z, \qquad w_2 = 1, \qquad A = 2 + |z|^2, \qquad J = 0. \tag{5.4}$$

Using the CP^2 representation (4.42) we can find that the associated CMC-surface is immersed in $\mathbb{R}^3$ and is given in a polar coordinates $\left(r = \left(x^2 + y^2\right)^{1/2}, \varphi\right)$ by

$$\begin{aligned} &X^1 = -2X^2 = 2X^6 = 2\sqrt{2}\left(2 + r^2\right)^{-1}, \qquad X^3 = -X^7 = -2r\left(2 + r^2\right)^{-1}\sin\varphi, \\ &X^5 = 0, \qquad X^4 = X^8 = 2r\left(2 + r^2\right)^{-1}\cos\varphi. \end{aligned} \tag{5.5}$$

The metric is conformally flat

$$I = \frac{2}{\left(2 + r^2\right)^2}\left(dr^2 + r^2 d\varphi^2\right) \tag{5.6}$$

This case corresponds to the immersion of the CP^2 model into the CP^1 model.

2. Another class of two-soliton solutions of the CP^2 model (4.6)–(4.8) is determined, for example, by two analytic functions; i.e. we can take, for example:

$$w_1 = z^2, \qquad w_2 = \sqrt{2}z, \qquad A = \left(1 + |z|^2\right)^2, \qquad J = 0. \tag{5.7}$$

Integrating formulae (4.42) we obtain the associated CMC-surface which can be written in a polar coordinates as follows

$$\begin{aligned} &X^1 = 2\left(1 + r^2\right)^{-2}, \qquad X^2 = -2\left(1 + 2r^2\right)\left(1 + r^2\right)^{-2}, \\ &X^3 = -2r^2\left(1 + r^2\right)^{-2}\sin 2\varphi, \qquad X^4 = 2r^2\left(1 + r^2\right)^{-2}\cos 2\varphi, \\ &X^5 = -2\sqrt{2}r\left(1 + r^2\right)^{-2}\sin\varphi, \qquad X^6 = 2\sqrt{2}r\left(1 + r^2\right)^{-2}\cos\varphi, \\ &X^7 = \sqrt{2}\left(2r^2 - 3\right)r\left(1 + r^2\right)^{-2}\sin\varphi, \\ &X^8 = \sqrt{2}\left(2r^2 - 3\right)r\left(1 + r^2\right)^{-2}\cos\varphi, \end{aligned} \tag{5.8}$$

The corresponding first fundamental form is conformal

$$I = \frac{2}{\left(1 + r^2\right)^2}\left(dr^2 + r^2 d\varphi^2\right). \tag{5.9}$$

3. A class of mixed solutions of the CP^2 model (4.2) is represented by

$$w_1 = \frac{\bar{z}+z}{1-|z|^2}, \qquad w_2 = \frac{\bar{z}-z}{1-|z|^2}, \qquad A = \left(\frac{1+|z|^2}{1-|z|^2}\right)^2, \qquad J = 0. \tag{5.10}$$

From (4.32) and using (5.2) we obtain the expression for the associated surface which can be written in polar coordinates as follows

$$\begin{aligned} &X^1 = X^2 = X^4 = X^5 = X^7 = 0, \\ &X^3 = \frac{-4r}{1+r^2}\sin\varphi, \qquad X^6 = \frac{-4r}{1+r^2}\cos\varphi, \qquad X^8 = \frac{4}{1+r^2} \end{aligned} \tag{5.11}$$

Hence this CMC-surface is really immersed in $\mathbb{R}^3$. The metric is conformal

$$I = \frac{16}{(1+r^2)^2}\left(dr^2 + r^2 d\varphi^2\right). \tag{5.12}$$

This case corresponds to the immersion of the CP^2 model into the CP^1 model.

If we take a more complicated example of this class, say,

$$\begin{aligned} &w_1 = \frac{-\bar{z}\left(3+2|z|^2\right)}{3-|z|^4}, \qquad w_2 = \frac{\sqrt{3}z\left(2+|z|^2\right)}{3-|z|^4}, \\ &A = \frac{\left(1+|z|^2\right)\left(|z|^6+6|z|^4+12|z|^2+9\right)}{\left(3-|z|^4\right)^2}, \qquad J = 0. \end{aligned} \tag{5.13}$$

then the expressions become very complicated but, in this case, we do have a genuine CP^2 solution.

4. An interesting class of meron-like solutions of the CP^1 model (3.5) is given by

$$w = \left(\frac{z}{\bar{z}}\right)^\beta, \qquad A = 2, \qquad J = \frac{-\beta^2}{2z^2}. \tag{5.14}$$

Here β can be an integer of a half-integer, with 2β being the meron number. Note also that all merons are located at $z = 0$ and so this solution is defined on $\mathbb{R}^2\backslash\{0\}$.

Then using the transformation (3.7) we find that the solution of the GW system (3.1) is given by

$$\psi_1 = \frac{\sqrt{\beta}}{2\sqrt{\bar{z}}}\left(\frac{z}{\bar{z}}\right)^{\frac{1}{2}}, \qquad \psi_2 = \frac{\sqrt{\beta}}{2\sqrt{z}}\left(\frac{z}{\bar{z}}\right)^{\frac{1}{2}}, \qquad p = \frac{\beta}{2|z|}. \tag{5.15}$$

The associated surface is obtained by integrating formulae (3.20) and we find

$$X_1 = \frac{-3}{2}\sin 2\beta\varphi, \qquad X_2 = \frac{-3}{2}\cos 2\beta\varphi, \qquad X_3 = \frac{\beta}{2}\ln r. \tag{5.16}$$

Thus the CMC-surface is a cylinder which is covered 2β times. Of course, the Gauss and mean curvatures are

$$K = 0, \qquad H = 1. \tag{5.17}$$

We note that our procedure of using the 'multimeron' solution of the CP^1 model to construct our CMC-surface has effectively involved mapping $\mathbb{R}^2\backslash\{0\}$ onto the equator of the unit sphere (merons) and then turning this circle into an infinite cylinder (based on this circle). This was done by effectively 'undoing' the radial projection of the previous map. Hence the two singular points of the multi-meron configuration (5.14) i.e. ($z = 0$ and the point at ∞) have got mapped at the 'ends' of the cylinder.

6 Final remarks and future developments

In this paper we have demonstrated links between the CP^1 and CP^2 sigma models and Weierstrass representations for two-dimensional surfaces immersed into Euclidean spaces $\mathbb{R}^3$ and $\mathbb{R}^8$, respectively. These links enabled us to present an algorithm for the construction of CMC-surfaces immersed into $\mathbb{R}^n$. This new approach has been tested in Section 5. It has proved to be effective, as we were able to reproduce easily the known results which, before, were obtained by much more complicated procedures. Its potential for providing new meaningful results has been exhibited in the case of mixed solutions of the CP^2 sigma model leading to new interesting surfaces in $\mathbb{R}^8$.

The analytic method of the construction of the CMC-surfaces immersed into $\mathbb{R}^n$ presented here is limited by several assumptions. For example we have studied only CP^1 and CP^2 models defined over S^2 (i.e. to consider $J \neq 0$) The question the arises as to whether our approach can be extended to higher CP^N models and to Weierstrass systems describing surfaces immersed in multi-dimensional Euclidean and pseudo-Riemannian spaces. If this is the case our approach may provide new classes of solutions and consequently new classes of surfaces in these multi-dimensional spaces. Other requirement of the proposed method, worth investigating further is the CP^N models involving maps from $\mathbb{R}^2$ (not necessarly S^2). We can expect that taking $J \neq 0$ can broaden the applicability of our approach.

Finally, it is worth noting that the CMC-surfaces can be used “in reverse” to address certain physical problems. Namely, we sometimes know the analytical description of CMC-surfaces in a physical system for which analytic models are not fully developed. Using our approach we can, perhaps, select an appropriate sigma model corresponding to the given Weierstrass representation and characterise the class of equations describing the physical phenomena in question. This was attempted successfully for Weierstrass representation for CMC-surfaces in 3-dimensional Euclidean space [30], but not to our knowledge for multi-dimensional spaces. These and other questions will be addressed in future work.

Acknowledgements

AMG thanks A Strasburger (University of Warsaw) for helpful and interesting discussions on the topic of this paper. WJZ would like to thank the CRM, Universite de Montreal and Center for Theoretical physics, MIT for the support of his stay and their hospitality. AMG would like to thank the University of Durham for the award of Alan Richards fellowship to him that allowed him to spend two terms in Durham during the academic years 2001 and 2002. Partial support for AMG's work was provided also by a grant from NSERC of Canada and the Fonds FCAR du Quebec.

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